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Instantons in the double-tensor multiplet

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Abstract: The double-tensor multiplet naturally appears in type IIB superstring compactifications on Calabi-Yau threefolds, and is dual to the universal hypermultiplet. We revisit the calculation of instanton corrections to the low-energy effective action, in the supergravity approximation. We derive a Bogomolny'i bound for the double-tensor multiplet and find new instanton solutions saturating the bound. They are characterized by the topological charges and the asymptotic values of the scalar fields in the double-tensor multiplet.

Keywords: Nonperturbative Effects, Solitons Monopoles and Instantons, Supergravity Models.

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Contents

1. Introduction

Instanton effects in string and M theory are still relatively poorly understood. This is due to the lack of a conventional instanton calculus as we know it from (supersymmetric) field theory. A well-known open problem is to determine the instanton corrections to the hypermultiplet moduli space of type II superstrings or M theory compactified on a Calabi-Yau (CY) threefold down to four or five dimensions. Supersymmetry requires the hypermultiplet moduli space $\mathcal{M}_H$ to be quaternion-Kähler [1]. The four- (or five-) dimensional dilaton lives in a multiplet which can be dualized into the universal hypermultiplet. Hence, $\mathcal{M}_H$ receives quantum corrections, and the instantons correspond to euclidean p-branes wrapping $p+1$ cycles of the CY [2].

The simplest setup for studying this problem, is to consider CY-compactifications of M theory/type IIA superstrings with Hodge number $h_{2,1} = 0$, or, for type IIB, $h_{1,1} = 0$,[1] since this yields a low-energy effective action of $N = 2$ supergravity coupled to a single hypermultiplet, such that the moduli space $\mathcal{M}_H$ has dimension four. From a type IIB perspective, this hypermultiplet arises from dualizing the *double-tensor* multiplet, whose bosonic components descend from the NS-NS and R-R two-forms and scalars in ten dimensions. This suggests that instanton calculations should be done on the double-tensor multiplet side. In the next section, we shall make another argument, which also applies to type IIA and M theory, why the double-tensor multiplet is more appropriate for our purposes.

[1]By type IIB with $h_{1,1} = 0$ we mean the mirror version of IIA with $h_{1,2} = 0$. As explained in [3], this model has to be understood in terms of a Landau-Ginzburg description instead of a geometric compactification, since all CY manifolds are Kähler and have $h_{1,1} > 0$.

Yet, even in the case of a single hypermultiplet, it is difficult to compute instanton effects directly in string theory, without explicit knowledge of the instanton measure and the details of the wrapped branes along the CY cycles. Therefore, we will study this problem in a pure supergravity context, in which semi-classical instanton calculations can be done in the more conventional and "field-theoretic" way, following a similar strategy as in [4, 5], or as in [6] for matter coupled to $N = 1$ supergravity. Although being an approximation of the exact result, the hope is that the leading supergravity corrections, combined with the constraints from quaternion-Kähler geometry, and together with some knowledge from string theory on the isometries and singularity structure of $\mathcal{M}_H$, should fix the answer uniquely. Such a program has worked succesfully in the context of supersymmetric field theories in three dimensions with eight supercharges, where the hypermultiplet moduli space is hyperkähler [7, 8]. See [9, 10] for related issues.

In this paper, we carry out the first steps of the supergravity instanton calculation. In section 2, we explain how the euclidean theory is best understood in terms of the double-tensor multiplet, since then the action is manifestly positive definite, a requirement needed for a semiclassical approximation. In section 3, we derive a Bogomolny'i bound and show that the instanton action is purely topological and given by a surface term. We then solve the BPS equation explicitly and compute the instanton action for the solutions. A similar approach was followed in [4] and [5]. Compared to these papers, we propose a different euclidean version of the universal hypermultiplet lagrangian, so our results, where comparable, are somehow different. Moreover, we have found new instanton solutions, which will play an important role in understanding the quantum corrected hypermultiplet moduli space, as explained in the discussion at the end of the paper.

2. The double-tensor multiplet

As mentioned in the introduction, we are interested in the case of a single hypermultiplet coupled to $N = 2$ supergravity. Classically, the four scalars of the universal hypermultiplet parametrize the homogeneous quaternion-Kähler manifold [11, 12]

$$\mathcal{M}_H = \frac{\mathrm{SU}(1,2)}{\mathrm{U}(2)}\,. \tag{2.1}$$

In a basis of real fields $\{\phi, \chi, \varphi, \sigma\}$, the bosonic lagrangian takes the form[2]

$$\mathcal{L}_{\mathrm{UH}} = -\mathrm{d}^D x\,\sqrt{g}\,R + \frac{1}{2}|\mathrm{d}\phi|^2 + \frac{1}{2}\,\mathrm{e}^{-\phi}\left(|\mathrm{d}\chi|^2 + |\mathrm{d}\varphi|^2\right) + \frac{1}{2}\,\mathrm{e}^{-2\phi}\,|\mathrm{d}\sigma + \chi \mathrm{d}\varphi|^2\,, \tag{2.2}$$

with $D = 4$ or 5, depending on whether one is interested in type II or M theory compactifications. The lagrangian has a global $\mathrm{SU}(1,2)$ isometry group.

For our purposes, it will be convenient to discuss the dual version of $\mathcal{L}_{\mathrm{UH}}$ in terms of a double-tensor multiplet. Consider the first-order lagrangian

$$\mathcal{L}_{\mathrm{DT}} = -\mathrm{d}^D x\,\sqrt{g}\,R + \frac{1}{2}|\mathrm{d}\phi|^2 + \frac{1}{2}\,\mathrm{e}^{-\phi}|\mathrm{d}\chi|^2 + \frac{1}{2}M_{ab}\,{*H^a} \wedge H^b - \lambda_a\,\mathrm{d}H^a\,, \tag{2.3}$$

[2]Throughout this paper, we use form notation. Lagrangians are written as volume forms, and we use the notation $|\omega_p|^2 \equiv *\omega_p \wedge \omega_p$.

where the H^a are a doublet of $(D-1)$ forms, the λ_a are two scalars, and

$$M = \mathrm{e}^{\phi} \begin{pmatrix} 1 & -\chi \\ -\chi & \mathrm{e}^{\phi} + \chi^2 \end{pmatrix} . \tag{2.4}$$

The two scalars ϕ and χ parametrize the coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{O}(2)$; in terms of the complex combination

$$\tau \equiv \chi + 2\mathrm{i}\,\mathrm{e}^{\phi/2} \tag{2.5}$$

the scalar part of $\mathcal{L}_{\mathrm{DT}}$ can be written as $2|d\tau/\,\mathrm{Im}\,\tau|^2$. The tensor terms, however, break the global $\mathrm{SL}(2,\mathbb{R})$ symmetry, leaving only shift symmetries of ϕ and χ. The shift in χ acts as

$$\tau \to \tau + b\,, \qquad \begin{pmatrix} H^1 \\ H^2 \end{pmatrix} \to \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} H^1 \\ H^2 \end{pmatrix}, \qquad \begin{pmatrix} \lambda^1 \\ \lambda^2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} \begin{pmatrix} \lambda^1 \\ \lambda^2 \end{pmatrix}, \tag{2.6}$$

whereas the shift in ϕ acts as

$$\tau \to \mathrm{e}^{\kappa}\tau\,, \qquad \begin{pmatrix} H^1 \\ H^2 \end{pmatrix} \to \begin{pmatrix} \mathrm{e}^{-\kappa} H^1 \\ \mathrm{e}^{-2\kappa} H^2 \end{pmatrix}, \qquad \begin{pmatrix} \lambda^1 \\ \lambda^2 \end{pmatrix} \to \begin{pmatrix} \mathrm{e}^{\kappa}\lambda^1 \\ \mathrm{e}^{2\kappa}\lambda^2 \end{pmatrix} . \tag{2.7}$$

Note that the latter acts like an $\mathrm{SL}(2,\mathbb{R})$ transformation on τ, but not on the H^a. The full type IIB theory compactified to four dimensions (classically) has $\mathrm{SL}(2,\mathbb{R})$ symmetry, due to the presence of additional tensor multiplets which transform nontrivially [13]. Setting the scalars in these multiplets to nonvanishing constants results in a breakdown of the symmetry and leaves only the above transformations as residual invariances.

The equations of motion for the Lagrange multipliers λ_a imply that the H^a are closed. Writing $H^a = \mathrm{d}B^a$, one obtains the double-tensor multiplet. Integrating out the tensors instead gives the duality relation

$$\mathrm{d}\lambda_a = -M_{ab} * H^b . \tag{2.8}$$

Substituting this back yields the action for the universal hypermultiplet (2.2), upon identifying

$$\lambda_1 = \varphi\,, \qquad \lambda_2 = \sigma\,. \tag{2.9}$$

The dual formulation in terms of the double-tensor multiplet is not unique.[3] We can start with (2.3), but write everywhere φ instead of χ. Dualizing the tensors and identifying

$$\lambda_1 = -\chi\,, \qquad \lambda_2 = \sigma + \varphi\,\chi \tag{2.10}$$

yields the same hypermultiplet action, as one can easily check.

[3]This fact was already observed in [14]. The reason is that one can choose inequivalent commuting $\mathrm{U}(1)\times\mathrm{U}(1)$ factors in the isometry group $\mathrm{SU}(1,2)$ with respect to which we dualize the universal hypermultiplet into the double-tensor multiplet. The choices above correspond to shifts in φ and σ, and a shift in σ together with the transformation $\chi \to \chi + \epsilon$, $\sigma \to \sigma - \epsilon\varphi$.

In addition, the dualization procedure yields a boundary term which has to be added to the hypermultiplet action,

$$\mathcal{L}_{\rm bnd} = (-)^D \, {\rm d}\big[\lambda_a \, (M^{-1})^{ab} * {\rm d}\lambda_b\big] \, , \tag{2.11}$$

where we used that, when acting on a p-form in Minkowski space, $** = -(-)^{(D-1)p}$. The different choices corresponding to (2.9) and (2.10) would now give different boundary terms. However, due to the isometries of the scalar manifold, they are related to each other by a field redefinition of the multipliers, $\tilde{\sigma} = \sigma + \varphi\chi$, $\tilde{\varphi} = -\chi$, $\tilde{\chi} = \varphi$. Substituting (2.10) into (2.11), we get

$$\mathcal{L}_{\rm bnd} = (-)^D \, {\rm d}\big[{\rm e}^{-\phi}\chi *{\rm d}\chi + {\rm e}^{-2\phi}\sigma *({\rm d}\sigma + \chi{\rm d}\varphi)\big] \, . \tag{2.12}$$

The total action for the universal hypermultiplet is then

$$\mathcal{L} = \mathcal{L}_{\rm UH} + \mathcal{L}_{\rm bnd} \, . \tag{2.13}$$

The fermions have been suppressed here. For hypermultiplets, the supersymmetry transformation rules and the fermion-terms in the lagrangian are known in general. For the double-tensor multiplet lagrangian (2.3), the fermion-terms and susy rules can be determined by dualization. However, the most general self-interacting supersymmetric double-tensor multiplet lagrangian has not been worked out. For a discussion on this in the context of rigid $N = 2$ supersymmetry, we refer to [15].

Euclidean formulation. To find instanton solutions, we need the euclidean formulation of the universal hypermultiplet, or, equivalently, the euclidean double-tensor multiplet lagrangian. For the latter, apart from the usual complications with the euclidean Einstein-Hilbert term, the Wick rotation acts in the standard way on the scalars and tensors. While the double-tensor multiplet lagrangian formally stays the same,

$$\mathcal{L}^E_{\rm DT} = {\rm d}^D x \, \sqrt{g} R + \frac{1}{2}|{\rm d}\phi|^2 + \frac{1}{2}\,{\rm e}^{-\phi}|{\rm d}\chi|^2 + \frac{1}{2} M_{ab} *H^a \wedge H^b \, , \tag{2.14}$$

the dual euclidean universal hypermultiplet lagrangian has two sign flips in the kinetic terms, due to the fact that we now have $** = (-)^{(D-1)p}$ when acting on a p-form in euclidean space. The dualization procedure yields

$$\mathcal{L}^E_{\rm UH} = {\rm d}^D x \, \sqrt{g} R + \frac{1}{2}|{\rm d}\phi|^2 + \frac{1}{2}\,{\rm e}^{-\phi}\left(|{\rm d}\chi|^2 - |{\rm d}\varphi|^2\right) - \frac{1}{2}\,{\rm e}^{-2\phi}\,|{\rm d}\sigma + \chi {\rm d}\varphi|^2 \, , \tag{2.15}$$

together with the boundary term

$$\mathcal{L}^E_{\rm bnd} = -(-)^D \, {\rm d}\big[{\rm e}^{-\phi}\chi *{\rm d}\chi + {\rm e}^{-2\phi}\sigma *({\rm d}\sigma + \chi{\rm d}\varphi)\big] \, . \tag{2.16}$$

By setting $\varphi = \chi = 0$, this boundary term is the same as for the D-instanton of type IIB in ten dimensions, obtained by dualizing the nine-form field strength into the R-R scalar σ [16, 17]. In four dimensions, we generate more terms due to the fact that we dualize two tensors. The sign flips of the kinetic terms of the two dual fields λ_a are compatible with the prescription of Wick rotating pseudoscalars $\lambda_a \to {\rm i}\lambda_a$ [18]. This is consistent with the duality relation (2.8).

A euclidean version of the universal hypermultiplet action was also proposed in [5]. Both their bulk lagrangian and boundary term differ from ours. This has important consequences since the instanton action defines the weight in the path integral, and hence correlation functions and eventually the quantum-corrected hypermultiplet moduli space will be different.

Due to the sign changes in (2.15), the geometry of the scalar manifold is no longer $\mathrm{SU}(1,2)/\,\mathrm{U}(2)$. Instead, it is given by the coset space

$$\mathcal{M}_H^E = \frac{\mathrm{SL}(3,\mathbb{R})}{\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(1,1)}\,, \tag{2.17}$$

which is *not* a quaternion-Kähler manifold. This is not in contradiction with supersymmetry, since only *minkowskian* supersymmtry requires the target space to be quaternionic [1]. A brief discussion on the geometry of the space (2.17) is given in appendix A.

In four dimensions, the same target space can be obtained by applying the **c**-map [11] to pure $N = 2$ euclidean supergravity [19]. This turns the four bosonic degrees of freedom contained in the metric and graviphoton into the four scalars of the universal hypermultiplet and gives rise to the two sign flips. Moreover, the **c**-map maps Reissner-Nordstrom black hole solutions to D-instantons in the universal hypermultiplet, as was shown in [20].

We remark that it is the inverted signs in the euclidean hypermultiplet action that make instanton solutions in flat space possible. Indeed, the trace of the Einstein equation sets the bulk lagrangian to zero, hence nontrivial field configurations would require a nonvanishing curvature scalar if the sigma model part of the lagrangian were positive definite. The negative signs in (2.15) allow for cancellations that are compatible with $R = 0$. Note also that since on the hypermultiplet side the bulk action vanishes for any solution, the instanton action comes entirely from the boundary term discussed above. As already stated, the boundary term (2.16) is different from the one proposed in [5]. For this reason, we get different results for the instanton action, and eventually for the instanton corrected hypermultiplet moduli space.

What is more important from the point of view of instanton calculations, is that the euclidean lagrangian (2.15) is no longer positive definite. In a path integral formulation, this makes the finite action configurations irrelevant, since the action is not bounded from below. Moreover, perturbative fluctuations around the instanton yield diverging non-gaussian integrals, and the semiclassical approximation would break down. Similar considerations apply to the $N = 2$ tensor multiplet, whose euclidean action is not positive definite.

On the other hand, the euclidean double-tensor multiplet lagrangian is bounded from below, since the matrix M_{ab} is positive definite. This leads to a well-defined semiclassical treatment, in which the instantons dominate the euclidean path integral. For this reason, it is important to perform all calculations on the double-tensor multiplet side, and after having computed the instanton corrections there, we can dualize to the hypermultiplet formulation. As a consequence, the quantum corrected quaternion-Kähler geometry will have at least two commuting $\mathrm{U}(1)$ isometries. Such metrics were classified in [14, 21].

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3. Instanton solutions

3.1 Asymptotics

Before finding the explicit instanton solutions, it will be useful to discuss the asymptotic behaviour of the fields that can lead to a finite action. Since the euclidean action (2.14) consists of three positive definite terms, each term individually should integrate to a finite quantity. For simplicity we consider for the moment flat four-dimensional space. This determines the following behaviour at infinity:

$$\phi \to \phi_\infty + \mathcal{O}\left(\frac{1}{r^2}\right) , \qquad \chi \to \chi_\infty + \mathcal{O}\left(\frac{1}{r^2}\right) , \qquad H_{\mu\nu\rho} \propto \frac{1}{r^3} . \tag{3.1}$$

The asymptotic value of ϕ is identified with the four- (or five-) dimensional string coupling constant,

$$g_s \equiv \mathrm{e}^{-\phi_\infty/2} . \tag{3.2}$$

The field strengths determine topological charges, defined by integrating the tensors H^a over spheres at infinity,

$$\int_{S_\infty^{D-1}} H^a = Q^{(a)} , \qquad a = 1, 2 . \tag{3.3}$$

In the dual (hypermultiplet) formulation, topological charges become Noether charges, corresponding to the Peccei-Quinn symmetries which act as constant shifts in the Lagrange multipliers λ_a. These charges descend from the brane charges in ten or eleven dimensions, and, in the appropriate units, are expected to be quantized.

The euclidean space we shall concentrate on is actually flat space with a countable number of points, the locations of the instantons, excised,[4]

$$\mathcal{M} = \mathbb{R}^D - \cup_i \{\vec{x}_i\} , \tag{3.4}$$

such that non-trivial cycles with corresponding charges (3.3) exist. Stated differently, in the supergravity approximation it will typically not be possible to find regular solutions at the locations of the instantons, as we will explicitly see below. The only singularity which can still lead to a finite action is a logarithmic singularity in ϕ at the origin,

$$\phi \to c \ln r , \tag{3.5}$$

for some constant c. In our examples below, χ will tend to a constant χ_0, and the tensors have the same $1/r^3$ behaviour such that the charges stay the same when the H^a are integrated around an infinitesimal sphere around the origin.

3.2 The Bogomolny'i bound

The euclidean double-tensor multiplet action (2.14) is positive semi-definite (apart from the Einstein-Hilbert term). In fact, we can derive a lower bound by writing it as

$$\mathcal{L}^E_{\mathrm{DT}} = \mathrm{d}^D x \sqrt{g} R + \frac{1}{2} *(N{*}H + OE)^t \wedge (N{*}H + OE) + (-)^D H^t \wedge N^t OE . \tag{3.6}$$

[4] A possible contribution to the action from a Gibbons-Hawking boundary term will then be absent.

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Here we have defined

$$H = \begin{pmatrix} H^1 \\ H^2 \end{pmatrix}, \qquad E = \begin{pmatrix} \mathrm{d}\phi \\ \mathrm{e}^{-\phi/2}\,\mathrm{d}\chi \end{pmatrix}, \qquad N = \mathrm{e}^{\phi/2} \begin{pmatrix} 0 & \mathrm{e}^{\phi/2} \\ 1 & -\chi \end{pmatrix}, \tag{3.7}$$

such that $N^t N = M$, and O is some orthogonal (scalar) field-dependent matrix, whose appearance is due to the fact that N and the zweibein E are determined only modulo local O(2) transformations.

Clearly, the action is bounded from below by

$$S^E \geq \int_{\mathcal{M}} \left(\mathrm{d}^D x \sqrt{g} R + (-)^D H^t \wedge N^t O E \right) , \tag{3.8}$$

where the second term is topological, as it is independent of the spacetime metric. The bound is saturated by field configurations satisfying the BPS condition

$$*H = -N^{-1} O E \,. \tag{3.9}$$

A similar Bogomolny'i equation was derived for an $N = 1$, $D = 4$ tensor multiplet (containing one tensor and one scalar) in [6]. Notice that, if the matrix O is invariant, this equation transforms covariantly under (2.6) and (2.7).

Equation (3.9) is a proper BPS condition only if it implies the equations of motion, and this will fix the O(2) degeneracy. It is easily verified that field configurations satisfying (3.9) have vanishing energy-momentum tensors, hence they can exist only in Ricci-flat spaces. We therefore have to amend our BPS condition by the equation $R_{\mu\nu}(g) = 0$.

For the field equations of the tensors, $\mathrm{d}(M * H) = 0$, to hold we must have

$$\mathrm{d}(N^t O E) = 0 \,. \tag{3.10}$$

This condition also guarantees that the topological term in (3.8) is closed and hence can locally be written as a total derivative. As a consequence, it does not contribute to the equations of motion such that also the field equations for the scalars are guaranteed to be satisfied. The latter follow from requiring that the solution of (3.9) correspond to closed forms for H^a,

$$\mathrm{d}(N^{-1} O * E) = 0 \,. \tag{3.11}$$

To determine the O(2) matrices that are compatible with (3.10), we parametrize O by

$$O = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \begin{pmatrix} c & -s \\ s & c \end{pmatrix}, \tag{3.12}$$

where the functions $c(\phi, \chi)$ and $s(\phi, \chi)$ are constrained by $c^2 + s^2 = 1$, and $\epsilon = \pm 1$ for the two components of O(2) with $\det O = \epsilon$. Equation (3.10) then gives rise to the differential equations

$$\begin{aligned} 0 &= \partial_\phi c - \mathrm{e}^{\phi/2} \partial_\chi s \\ 0 &= \partial_\phi s + \mathrm{e}^{\phi/2} \partial_\chi c - \frac{1}{2}(2\epsilon - 1) s \,. \end{aligned} \tag{3.13}$$

We derive the general solution in appendix B. The result is that there are *three* distinct BPS conditions corresponding to the O(2) matrices

$$O_{1,2} = \pm \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix}, \qquad O_3 = \pm \frac{1}{|\tau'|} \begin{pmatrix} \mathrm{Re}\,\tau' & -\mathrm{Im}\,\tau' \\ \mathrm{Im}\,\tau' & \mathrm{Re}\,\tau' \end{pmatrix}, \tag{3.14}$$

invariant under both (2.6) and (2.7). Here $\tau' = \tau - \chi_0$ with τ as in (2.5), χ_0 is a real integration constant, and the plus and minus signs refer to the instanton and anti-instanton, respectively.

For these three O(2) matrices the 1-form $N^t OE$ is exact,

$$N^t OE = \pm\, \mathrm{d}Y\,, \tag{3.15}$$

where modulo an additive constant

$$Y_{1,2} = \begin{pmatrix} \epsilon\chi \\ \mathrm{e}^{\phi} - \frac{1}{2}\epsilon\chi^2 \end{pmatrix}, \qquad Y_3 = \frac{1}{2}\sqrt{4\mathrm{e}^{\phi} + (\chi-\chi_0)^2}\begin{pmatrix} 2 \\ -\chi-\chi_0 \end{pmatrix}. \tag{3.16}$$

It follows that the action for BPS configurations is given by a topological boundary term

$$S^E|_{\mathrm{BPS}} = (-)^D \int_{\mathcal{M}} H^t \wedge N^t OE = \mp \int_{\partial\mathcal{M}} Y^t H\,. \tag{3.17}$$

The instanton action is therefore determined by the charges $Q^{(a)}$ and the values of the fields χ and e^{ϕ} at the boundaries.

It is easy to find the corresponding BPS equation in the dual hypermultiplet formulation. Using (2.8), (3.9), (3.15) and the fact that $M = N^t N$, we find for the Lagrange multipliers

$$\mathrm{d}\lambda = \pm\, \mathrm{d}Y\,, \tag{3.18}$$

such that, up to a constant, the solutions for the two extra scalars are completely determined in terms of ϕ and χ.

3.3 Solutions and instanton action

We can solve the BPS condition (3.9) for the three possible matrices O. For $O_1 = \pm 1$, the condition reads

$$*H = \pm \begin{pmatrix} \chi \overset{\leftrightarrow}{\mathrm{d}} \mathrm{e}^{-\phi} \\ \mathrm{d}\mathrm{e}^{-\phi} \end{pmatrix}. \tag{3.19}$$

Applying d$*$ to the equation and using the Bianchi identities of H, we find that $\mathrm{e}^{-\phi}$ must be harmonic, and from the first component it then follows that also χ satisfies the Laplace equation,

$$O_1: \quad \mathrm{d}*\mathrm{d}\mathrm{e}^{-\phi} = 0\,, \qquad \mathrm{d}*\mathrm{d}\chi = 0\,. \tag{3.20}$$

As mentioned above, scalars satisfying these conditions will also solve their field equations.

In the following, we consider for simplicity spherically symmetric configurations (single instantons) in flat space only. The dilaton equation of motion is then solved by

$$\mathrm{e}^{-\phi} = \mathrm{e}^{-\phi_\infty} + \frac{|Q^{(2)}|}{\Omega_D\, r^{D-2}}\,. \tag{3.21}$$

Here $\Omega_D = (D-2)\mathrm{Vol}(S^{D-1})$, and we have chosen the location of the instanton ($\vec{x}_1$ in (3.4)) as the origin. The integration constant $Q^{(2)}$ appearing in the solution is equal to the topological charge associated with H^2, as follows from the second equation of (3.19). The 'selfdual' instanton (upper sign in (3.19)) is taken for negative $Q^{(2)}$, the 'anti-selfdual' instanton for positive $Q^{(2)}$.

Since, up to proportionality factors, there is only a unique spherically symmetric harmonic function, χ must be of the form $\chi = \chi_1 \mathrm{e}^{-\phi} + \chi_0$ with χ_0, χ_1 constant. It then follows from (3.19) that χ_0 is determined by

$$\chi_0 = \frac{Q^{(1)}}{Q^{(2)}}\,, \tag{3.22}$$

and this relation is consistent with the shift symmetries (2.6) and (2.7), since the charges transform non-trivially. The instanton action for O_1 is given by

$$S_1^E = \mp \int_{\partial\mathcal{M}} \left[\chi H^1 + \left(\mathrm{e}^{\phi} - \frac{1}{2}\chi^2\right) H^2\right], \tag{3.23}$$

where the boundary consists of the disjoint union of two spheres, $\partial\mathcal{M} = S_\infty^{D-1} \cup S_0^{D-1}$, with radii as indicated. The terms involving χ will diverge on S_0^{D-1} since χ is harmonic, so in order to obtain a finite action we have to take $\chi = \chi_0$ constant. This was already anticipated from the asymptotic behaviour of the fields, discussed in the beginning of this section. The action then reads

$$S_1^E = \frac{|Q^{(2)}|}{g_s^2}\,. \tag{3.24}$$

This solution was also found in [4, 5], and should correspond to the fivebrane wrapping the entire Calabi-Yau [2]. The instanton action is positive and hence does not distinguish instantons from anti-instantons. Imaginary theta-angle-like terms will have to be added to make this distinction.

Turning to O_2, we have the BPS condition

$$*H = \pm\, \mathrm{d} \begin{pmatrix} \mathrm{e}^{-\phi}\chi \\ \mathrm{e}^{-\phi} \end{pmatrix}. \tag{3.25}$$

Again, $\mathrm{e}^{-\phi}$ is harmonic, and the same now applies to $\mathrm{e}^{-\phi}\chi$. If one imposes rotational symmetry then

$$O_2: \quad \mathrm{d}*\mathrm{d}\mathrm{e}^{-\phi} = 0\,, \qquad \chi = \chi_1 \mathrm{e}^{\phi} + \chi_0\,, \tag{3.26}$$

and from (3.25), it follows again that $Q^{(1)} = \chi_0 Q^{(2)}$. Notice that the field χ is now completely regular everywhere, and interpolates between the boundaries according to

$$\Delta\chi \equiv \chi_\infty - \chi_0 = \frac{\chi_1}{g_s^2}\,. \tag{3.27}$$

The complete solution agrees with the asymptotics derived in (3.1) and (3.5).

For this solution, with the dilaton again given by (3.21), the instanton action (3.17) then becomes

$$S_2^E = |Q^{(2)}| \left(\frac{1}{g_s^2} + \frac{1}{2}(\Delta\chi)^2\right). \tag{3.28}$$

For the particular case of $\Delta\chi = 0$, the solution and instanton action are the same as for the O_1 solution. Notice also that both terms are positive and invariant under the shift symmetries (2.6) and (2.7), as guaranteed by the properties of the original action. For $\Delta\chi \neq 0$, our instanton solution is new, and this term in the instanton action does not depend on the string coupling constant g_s. The appearence of $\Delta\chi$ in the instanton action is one of the new results in this paper. Its presence was somehow anticipated in [2], and here we have computed it explicitly.

We now turn to O_3. The BPS equation for this case reads

$$*H = \pm\frac{1}{|\tau'|}\begin{pmatrix} -2\,\mathrm{d}\phi + \mathrm{e}^{-\phi}(\chi+\chi_0)\,\mathrm{d}\chi + \chi(\chi-\chi_0)\,\mathrm{d}\mathrm{e}^{-\phi} \\ (\chi-\chi_0)\,\mathrm{d}\mathrm{e}^{-\phi} + 2\mathrm{e}^{-\phi}\mathrm{d}\chi \end{pmatrix}. \tag{3.29}$$

Let us consider two Ansätze for which we can explicitly solve the equations. First, we set $\chi = 2\chi_1\mathrm{e}^{\phi/2} + \chi_0$. Then the equations simplify to

$$*H = \pm 2\,\mathrm{d}\mathrm{e}^{-\phi/2}\begin{pmatrix} \sqrt{1+\chi_1^2} \\ 0 \end{pmatrix}. \tag{3.30}$$

It follows that now $\mathrm{e}^{-\phi/2}$ is harmonic, with solution

$$\mathrm{e}^{-\phi/2} = \mathrm{e}^{-\phi_\infty/2} + \frac{|Q^{(1)}|}{2\sqrt{1+\chi_1^2}\,\Omega_D\, r^{D-2}}. \tag{3.31}$$

The scalar χ is then completely regular and interpolates between the boundaries as

$$\Delta\chi = \chi_\infty - \chi_0 = \frac{2\chi_1}{g_s}. \tag{3.32}$$

Since $H^2 = 0$ we have $Q^{(2)} = 0$, and for the instanton action we find

$$S_3^E = |Q^{(1)}|\sqrt{\frac{4}{g_s^2} + (\Delta\chi)^2} = |Q^{(1)}|\,|\tau'_\infty|, \tag{3.33}$$

where $\tau'_\infty = (\chi_\infty - \chi_0) + 2\mathrm{i}\,\mathrm{e}^{\phi_\infty/2}$ is the value of τ' at infinity. For $\Delta\chi = 0$, a similar solution was also found in [5]. Following the discussion in [2], it should correspond, from a IIA point of view, to the D2-brane wrapping a three-cycle in the Calabi-Yau, or to the D1+D3+D5-branes wrapping even cycles in type IIB. Notice again consistency with the symmetries (2.6) and (2.7). Observe also that for $\Delta\chi = 0$, the solution is inversely proportional to g_s, and is for small g_s dominating over the fivebrane instanton (3.28).

As a second Ansatz, consider $\chi = 2\chi_1\mathrm{e}^{\phi} + \chi_0$. This differs from the first Ansatz in the power of e^{ϕ}. The BPS condition turns into

$$*H = \pm 2\,\mathrm{d}\sqrt{\mathrm{e}^{-\phi} + \chi_1^2}\begin{pmatrix} 1 - \chi_1\chi_0 \\ -\chi_1 \end{pmatrix}. \tag{3.34}$$

Accordingly, the square root must be harmonic, and we find for $\chi_1 \neq 0$ (the case $\chi_1 = 0$ is included in the previous Ansatz),

$$\mathrm{e}^{-\phi} = (h-\chi_1)(h+\chi_1)\,, \qquad h = \sqrt{\mathrm{e}^{-\phi_\infty} + \chi_1^2} + \left|\frac{Q^{(2)}}{2\chi_1}\right| \frac{1}{\Omega_D\, r^{D-2}}. \tag{3.35}$$

The scalar field χ is regular everywhere and interpolates between zero and infinity as

$$\Delta\chi = \frac{2\chi_1}{g_s^2}\,. \tag{3.36}$$

The BPS equation further fixes the constant χ_1 to be

$$\chi_1 = -\frac{Q^{(2)}}{Q^{(1)} - \chi_0 Q^{(2)}}\,, \tag{3.37}$$

and the instanton action is easily computed from (3.17),

$$S_3^E = |\tau'_\infty| \left(|\hat{Q}^{(1)}| + \frac{1}{2}|\Delta\chi\, Q^{(2)}| \right). \tag{3.38}$$

We have redefined the $Q^{(1)}$ charge according to

$$\hat{Q}^{(1)} \equiv Q^{(1)} - \chi_0 Q^{(2)}\,, \tag{3.39}$$

such that it is invariant under (2.6). For $Q^{(2)} = 0$, the instanton action then clearly reduces to (3.33).

We have been unable to show that these two Ansätze yield the most general solution. The obtained results for the instanton action carry over to the hypermultiplet side, because the dualization procedure does not affect the value of the instanton action. As a consequence, the action does not depend on the asymptotic values of the two dual scalars λ_a. This is consistent with the $\mathrm{U}(1) \times \mathrm{U}(1)$ isometries of the universal hypermultiplet moduli space metric.

4. Discussion

In this paper, we have carried out the first steps of calculating instanton corrections to the hypermultiplet moduli space. An important ingredient was to derive a Bogomolny'i bound for the double-tensor multiplet lagrangian, and to solve the corresponding BPS equation. In a supersymmetric formulation, adapted to euclidean space, we expect our instanton solutions to preserve one half of the supersymmetries. A more general formulation for the double-tensor multiplet lagrangian, including the fermions and supersymmetry transformation rules, is presently under study. This will be important for finding the fermionic zero modes and eventually for computing instanton corrections to the relevant correlation functions that determine the hypermultiplet quantum-geometry. The exact moduli space must be consistent with the results derived in our paper. In particular, our supergravity instanton solutions should match with the results obtained from wrapping branes in the full ten-dimensional string theory. Stated differently, the universal hypermultiplet metric must contain exponential corrections which, at leading order in the string coupling constant and α', agree with the form of our instanton action. Using some results about quaternionic geometry [21, 14], it should be possible to find quaternionic metrics which asymptotically reproduce our results. We intend to report further on these issues in the near future.

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A. $\mathrm{SL}(3,\mathbb{R})/\,\mathrm{SL}(2,\mathbb{R})\times \mathrm{SO}(1,1)$

In this appendix, we discuss some geometrical aspects related to the sigma model corresponding to (2.15), with target space (2.17). The easiest way to study this space is by using the fact that the minkowskian version of the universal hypermultiplet moduli space is both Kähler and quaternion-Kähler. Since Kähler geometry is simpler to analyze, we will study the coset (2.17) from the point of view of Kähler geometry. Because of the sign flips compared to the minkowskian version, the target space will no longer be Kähler. It is therefore not possible to define complex coordinates together with a Kähler potential that determines the metric. As we show below, it is still possible to define coordinates and a potential from which the metric can be computed. To see this, we first define the fields

$$a = \sigma + \frac{1}{2}\chi\varphi\,, \qquad C_\pm = \frac{1}{2}(\varphi \pm \chi)\,, \tag{A.1}$$

in terms of which the sigma model part of the euclidean lagrangian (2.15) reads

$$\mathcal{L}^E_{\rm UH} = \frac{1}{2}|\mathrm{d}\phi|^2 - 2\mathrm{e}^{-\phi} *\mathrm{d}C_+ \wedge \mathrm{d}C_- - \frac{1}{2}\mathrm{e}^{-2\phi}\,|\mathrm{d}a + C_+ \overleftrightarrow{\mathrm{d}} C_-|^2\,. \tag{A.2}$$

If we further pass to coordinates $u_\pm^1, u_\pm^2 \in \mathbb{R}$ via the relations

$$S_\pm = \mathrm{e}^\phi \mp a - C_+C_- = \frac{1\mp u_\pm^1}{1 \pm u_\pm^1}\,, \qquad C_\pm = \frac{u_\pm^2}{1\pm u_\pm^1}\,, \tag{A.3}$$

then the lagrangian can be written as

$$\mathcal{L}^E_{\rm UH} = 2 g_{ij} *\mathrm{d}u_+^i \wedge \mathrm{d}u_-^j \tag{A.4}$$

with a metric

$$g_{ij} = -\frac{\partial^2}{\partial u_+^i \partial u_-^j}\,\ln\!\big(1 + u_+^1 u_-^1 + u_+^2 u_-^2\big)\,. \tag{A.5}$$

The $u_\pm^i$ are inhomogeneous coordinates of the coset space (2.17), transforming under $M \in \mathrm{SL}(2,\mathbb{R})$ as $u_+ \to M u_+$, $u_- \to (M^{-1})^t u_-$. We have therefore identified a potential in terms of real coordinates which determines the metric components. Such spaces are called para-Kähler.[5] The metric for the minkowskian universal hypermultiplet moduli space (2.1) is of the same form as in (A.5), but with $u_\pm^i$ treated as complex coordinates, where $u_-^i = -\bar{u}_+^i$ under complex conjugation.

[5] A more general discussion on para-Kähler manifolds, in the context of euclidean supergravity coupled to vector multiplets, will be given in [22].

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B. Determination of $\mathrm{O}(2)$ matrices

We need to solve the differential equations

$$0 = \partial_\phi c - \mathrm{e}^{\phi/2}\partial_\chi s \tag{B.1}$$

$$0 = \partial_\phi s + \mathrm{e}^{\phi/2}\partial_\chi c - \frac{1}{2}(2\epsilon - 1)s\,, \tag{B.2}$$

where c and s are subject to the constraint $c^2 + s^2 = 1$. We first multiply (B.1) by s and (B.2) by c, respectively, and use $-s\partial s = c\partial c$ to write the equations as

$$0 = s\,\partial_\phi c + \mathrm{e}^{\phi/2} c\,\partial_\chi c \tag{B.3}$$

$$0 = c\,\partial_\phi s + \mathrm{e}^{\phi/2} c\,\partial_\chi c - \frac{1}{2}(2\epsilon - 1)cs\,. \tag{B.4}$$

Multiplying the difference of these equations by c gives

$$0 = c\left[c\overset{\leftrightarrow}{\partial}_\phi s - \frac{1}{2}(2\epsilon - 1)cs\right] = \partial_\phi s - \frac{1}{2}(2\epsilon - 1)\,(1 - s^2)s\,, \tag{B.5}$$

which involves only s and can easily be integrated:

$$\frac{s^2}{1 - s^2} = \frac{1 - c^2}{c^2} = f^2(\chi)\,\mathrm{e}^{(2\epsilon - 1)\phi}\,. \tag{B.6}$$

The positive integration constant f^2 may depend on χ. These expressions we plug into the sum of (B.3) and (B.4),

$$\begin{aligned} 0 &= \mathrm{e}^{\phi/2}\partial_\chi c^2 + \partial_\phi(cs) - \frac{1}{2}(2\epsilon - 1)cs \\ &= -\frac{2f\,\mathrm{e}^{(3-4\epsilon)\phi/2}}{(f^2 + \mathrm{e}^{(1-2\epsilon)\phi})^2}\left[\partial_\chi f \pm \frac{1}{2}(2\epsilon - 1)\,\mathrm{e}^{(\epsilon - 1)\phi} f^2\right], \end{aligned} \tag{B.7}$$

where the sign ambiguity originates from taking the square root of $(cs)^2$. The equation is satisfied if the expression in square brackets vanishes. For $\epsilon = -1$, this is only possible if $f = 0$ since f is independent of ϕ. For $\epsilon = +1$, which corresponds to $O \in \mathrm{SO}(2)$, we find

$$f = 0 \qquad \text{or} \qquad f = \pm\frac{2}{\chi - \chi_0}\,, \tag{B.8}$$

with χ_0 an integration constant. $f = 0$ implies $c = \pm 1$ and $s = 0$. For nontrivial f we obtain (with the relative sign fixed by the original equations (B.1) and (B.2))

$$c = \pm\frac{\chi - \chi_0}{\sqrt{4\mathrm{e}^\phi + (\chi - \chi_0)^2}}\,, \qquad s = \pm\frac{2\,\mathrm{e}^{\phi/2}}{\sqrt{4\mathrm{e}^\phi + (\chi - \chi_0)^2}}\,, \tag{B.9}$$

or in terms of $\tau' = (\chi - \chi_0) + 2\mathrm{i}\,\mathrm{e}^{\phi/2}$,

$$c + \mathrm{i}s = \pm\frac{\tau'}{|\tau'|}\,. \tag{B.10}$$

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